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STABILITY LOSS DELAY IN A ZIEGLER SYSTEM†

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Stability loss delay in a Ziegler system is considered, on the assumption that the follower force is slowly increasing. An "input–output" function is constructed, enabling the time at which stability loss occurs to be predicted, given the initial magnitude of the force.

A delay in the loss of stability when the parameters are varied slowly was first observed for a model system [1]. A theory of the effect has been constructed for quite general systems [2–6]. Below we will consider the effect of a slow variation of the parameters on the dynamics of a modified Ziegler system [7]. Previous investigations of a Ziegler system using methods of classical bifurcation theory enabled many paradoxical features of non-conservative mechanical systems loaded by follower forces to be explained [8, 9].

We note that stability loss delay is not associated with the presence of any active controls that keep the system in the neighbourhood of an unstable steady state (cf. [10]).

1. DESCRIPTION OF THE SYSTEM. EQUATIONS OF MOTION

We define a Ziegler system to be a two-dimensional hinged-rod mechanical system with two degrees of freedom subject to a slowly increasing follower force **P** (Fig. 1). It is assumed that when the rods are horizontal the springs in the hinges are in their natural, undeformed, state. The rods, all of the same length l and bearing point masses of masses M and m, are assumed to be weightless.

As generalized coordinates of the system we use the angles φ_1 and φ_2 by which the rods deviate from the horizontal.

The dynamics of the system is described by the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\varphi_k}\right)-\frac{\partial T}{\partial \varphi_k}+\frac{\partial U}{\partial \varphi_k}=-\frac{\partial \Phi}{\partial \dot\varphi_k}+Q_k, \quad k=1,2$$

$$T=l^2\{(m+M)\dot\varphi_1^2+m\dot\varphi_2[2\cos(\varphi_1-\varphi_2)\dot\varphi_1+\dot\varphi_2]\}/2$$
$$U=U_g+U_e, \quad U_g=gl[(m+M)\sin\varphi_1+m\sin\varphi_2] \tag{1.1}$$
$$U_e=c\{\varphi_1^2(1+\delta\varphi_1^2)+(\varphi_1-\varphi_2)^2[1+\delta(\varphi_1-\varphi_2)^2]\}/2$$
$$\Phi=b[\dot\varphi_1^2+(\dot\varphi_1-\dot\varphi_2)^2]/2$$

where T is the kinetic energy of the system, U is its potential energy, which is the sum of the gravitational energy U_g and the energy of elastic deformation of the hinge springs U_e (c is the stiffness of the springs, δ is a parameter defining the non-linearity of the elastic properties of the springs, and g is the acceleration due to gravity), Φ is the dissipative function, which characterizes the dissipation of energy in the springs (b is the coefficient of dissipation), and Q_1 and Q_2 are generalized forces generated by the non-conservative follower force **P**.

The angle between the line of action of **P** and the horizontal is proportional, with coefficient χ, to the angle of deflection of the second rod ($0 < \chi \leqslant 1$). Follower forces of this type were considered, for example, in [11].

Formulae for the generalized forces may be found by computing the virtual work of **P**

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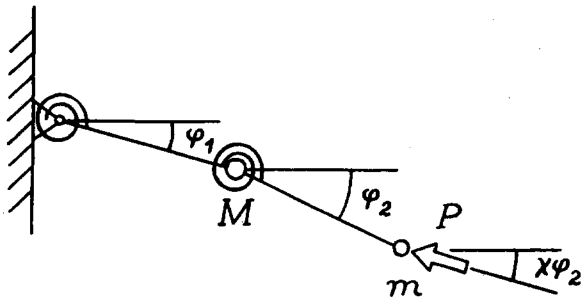

Fig. 1.

$$\delta A = (\mathbf{P}, \delta\mathbf{r}_m) = Q_1\delta\varphi_1 + Q_2\delta\varphi_2; \quad Q_k = \left(\mathbf{P}, \frac{\partial \mathbf{r}_m}{\partial \varphi_k}\right)$$

After reduction we obtain

$$Q_1 = -|\mathbf{P}|\sin(\chi\varphi_2 - \varphi_1), \quad Q_2 = -|\mathbf{P}|\sin(\chi - 1)\varphi_2$$

As an independent variable it is convenient to take dimensionless time $\tau = t/l(c/m)^{1/2}$. This enables us to rewrite the Lagrange equations (1.1) as follows:

$$\begin{aligned}&(1+\mu)\ddot\varphi_1 + \cos(\varphi_1-\varphi_2)\ddot\varphi_2 + \sin(\varphi_1-\varphi_2)\dot\varphi_1^2 + \beta(2\dot\varphi_1 - \dot\varphi_2) + \\ &+ 2\varphi_1 - \varphi_2 + 2\delta[\varphi_1^3 + (\varphi_1-\varphi_2)^3] + \varkappa(1+\mu)\cos\varphi_1 + p\sin(\chi\varphi_2 - \varphi_1) = 0 \\ &\cos(\varphi_1-\varphi_2)\ddot\varphi_1 + \ddot\varphi_2 - \sin(\varphi_1-\varphi_2)\dot\varphi_1^2 + \beta(\dot\varphi_2 - \dot\varphi_1) - \\ &-(\varphi_1-\varphi_2)[1+2\delta(\varphi_1-\varphi_2)^2] + \varkappa\cos\varphi_2 + p\sin(\chi-1)\varphi_2 = 0 \\ &(\mu = M/m, \quad \varkappa = lgm/c, \quad \beta = b/l(mc)^{1/2}, \quad p = |\mathbf{P}|l/c)\end{aligned} \tag{1.2}$$

The dots in (1.2) and below denote derivatives with respect to dimensionless time τ.

Remark. The term "Ziegler system" is generally used for a simpler mechanical system, obtained from the above by setting $\chi = 1$, $\delta = \varkappa = 0$ [7–9].

2. STABILITY LOSS DELAY IN A ZIEGLER SYSTEM

If the magnitude p of the dimensionless follower force is fixed, system (1.2) has a family of steady solutions

$$\begin{aligned}&\dot\varphi_1 = \dot\varphi_2 \equiv 0 \\ &\varphi_1 \equiv \varphi_{10}(\varkappa,\beta,\delta,\mu,\chi,p), \quad \varphi_2 \equiv \varphi_{20}(\varkappa,\beta,\delta,\mu,\chi,p)\end{aligned} \tag{2.1}$$

which, when $\varkappa \to 0$, tend to the trivial solution $\dot\varphi_1 = \dot\varphi_2 = \varphi_1 = \varphi_2 \equiv 0$.

If $p \leq p_{cr}(\varkappa, \beta, \delta, \mu, \chi)$, the solutions (2.1) are asymptotically stable; we will denote their domains of attraction in the phase space of the system by $\mathfrak{D}(\varkappa, \beta, \delta, \mu, \chi, p)$.

One can define in the parameter space $(\varkappa, \beta, \delta, \mu, \chi)$ domains S_1 and S_2 with the following properties: if the parameters lie in S_1, then when $p = p_{cr}$ one of the roots of the characteristic equation of system (1.2), linearized in the neighbourhood of the stationary solution (2.1), is zero. If the parameters lie in S_2, then when $p = p_{cr}$ a pair of roots of the characteristic equation lie on the imaginary axis.

Let us consider a Ziegler system with parameters in S_2 in which the magnitude of the follower force is slowly increased

$$p = p_i + \varepsilon\tau, \quad 0 < \varepsilon \ll 1, \quad 0 \leq p_i < p_{cr}(\varkappa,\beta,\delta,\mu,\chi) \tag{2.2}$$

In that case the phase trajectories of system (1.2) corresponding to initial conditions

$$(\dot{\varphi}_k(0), \varphi_k(0)) \in \mathcal{D}(\varkappa, \beta, \delta, \mu, \chi, p_i)$$

$$\left\{ \sum_{k=1}^{2} \dot{\varphi}_k^2(0) + (\varphi_k(0) - \varphi_{k0}(\varkappa, \beta, \delta, \mu, \chi, p_i))^2 \right\}^{1/2} \sim 1$$

will remain after the transient in an $O(\varepsilon)$-neighbourhood of the stationary solution of the instantaneous system (the system with fixed p equal to the latter's current value) until the follower force reaches a certain value p_0, which depends on p_i and exceeds p_{cr} by a quantity of order 1; when that occurs the system will break away from the steady solution—there will be a rapid development of oscillations [2–6]. Thus, the stability loss is "delayed": for a long time ($\sim \varepsilon^{-1}$) the phase trajectories will not leave a small neighbourhood of the unstable solution.

Numerical integration of Eqs (1.2) confirms the existence of a stability loss delay when the follower force is slowly increased.

As an example, Fig. 2 shows how the behaviour of one of the generalized coordinates of a Ziegler system with parameters $\varkappa = 0.1$, $\beta = 30.0$, $\delta = 1.5$, $\mu = 1.0$, $\chi = 0.8$ depends on the current value of the follower force (note that when $\varepsilon \neq 0$ the magnitude of the follower force may serve as "slow time" in system (1.2)). The computations were carried out for the following initial values of the follower force: 2.75, 2.25, 1.5, 0.5 (Fig. 2a–d). In all cases the parameter was taken as $\varepsilon = 10^{-3}$. The curve S in Fig. 2 is the value of φ_2 in the stationary solution (2.1) of the instantaneous system; the vertical dashed line indicates the critical value of the follower force $p_{cr} = 3.636$ at which the steady solution (2.1) destabilizes and a stable limit cycle is formed (Andronoff–Hopf bifurcation).

The computational results shown in Fig. 2, imply the following conclusion: the longer the trajectories are in a neighbourhood of a stable steady solution, the longer they will stay in the neighbourhood of the unstable solution. However, at a certain value of the follower force (≈ 5.5), oscillations will develop regardless of the time the phase trajectory spends in the neighbourhood of the stable solution.

In systems with parameters in S_1, there will be no delay of stability loss. Such systems will not be considered below.

3. INPUT–OUTPUT FUNCTION

We define the breakaway time to be the time τ at which the phase trajectory first leaves the ε^{σ}-neighbourhood of an unstable steady solution of the instantaneous system ($0 < \sigma < 1$).

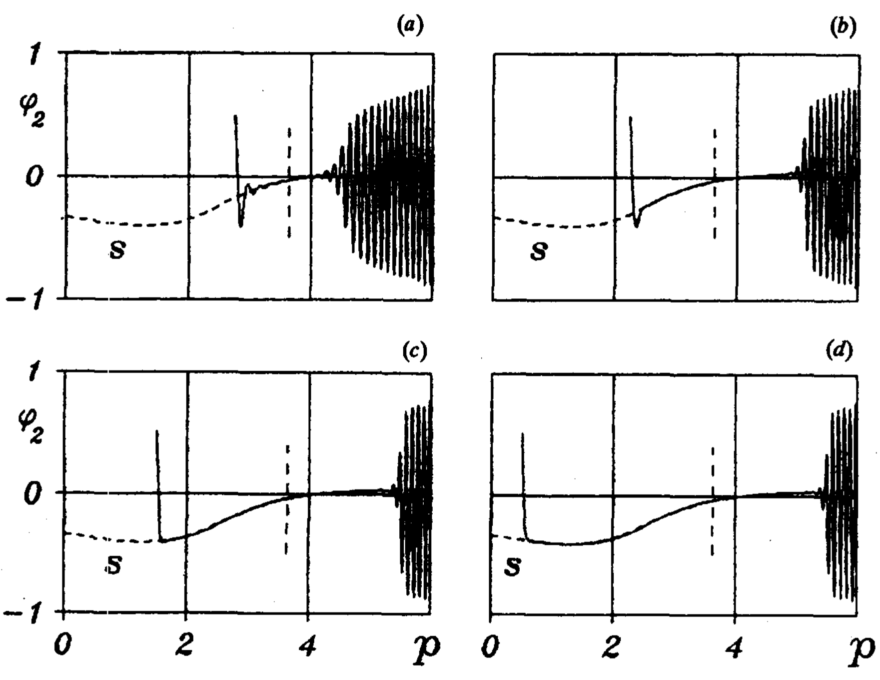

Fig. 2.

The input–output function $\Pi(\cdot)$ relates the value of the follower force at $\tau = 0$ and its limiting value (as $\varepsilon \to 0$) at the time the phase trajectory breaks away from the steady solution for most initial data $(\dot{\varphi}_k(0), \varphi_k(0)) \in \mathfrak{D}(\varkappa, \beta, \delta, \mu, \chi, p_i)$, such that

$$\left\{\sum_{k=1}^{2} \dot{\varphi}_k^2(0) + (\varphi_k(0) - \varphi_{k0}(\varkappa, \beta, \delta, \mu, \chi, p_i))^2\right\}^{1/2} > \varepsilon^{\sigma}$$

It should be noted that the limiting value does not depend on σ.

To construct the input–output function, one has to study certain properties of the roots of the characteristic equation of system (1.2), linearized in the neighbourhood of the steady solution of family (2.1), for complex values of the parameter p [4, 5].

Let $\lambda_1(p)$ be one of the two roots of the characteristic equation, which lie on the imaginary axis when $p = p_{cr}$. We will assume that $\operatorname{Im} \lambda(p_{cr}) < 0$.

We introduce the complex phase

$$\Psi(p) = \int_{p_{cr}}^{p} \lambda_1(p')dp' \tag{3.1}$$

In the general case, $\Psi(p)$ is a many-valued function of a complex variable. Its typical branch points will be points at which the root λ_1 vanishes or becomes multiple. We will consider the branch of $\Psi(p)$ obtained by analytic continuation from the real axis.

Bearing in mind the position of the level curves $\operatorname{Re} \Psi(p) = \text{const}$ in the complex p-plane, we will express the interval $I = [0, c_{cr}]$ as the union of an interval I_0 containing p_{cr}, defined as the set of all points of the interval on level curves of $\operatorname{Re} \Psi(p)$ that intersect the real axis twice (to the right and left of p_{cr}), and the interval $I_1 = I \backslash I_0$.

On I_0 the input–output function maps p onto the point $\Pi(p)$ on the real axis to the right of p_{cr} such that

$$\operatorname{Re}\Psi(p) = \operatorname{Re}\Psi(\Pi(p)) \tag{3.2}$$

If the independent variable in system (1.2) with $\varepsilon \neq 0$ is taken as the magnitude of the follower force, it can be proved, by analytic continuation along a path near the arc joining p and $\Pi(p)$ on a level curve of $\operatorname{Re} \Psi(p)$, that the solution for which the initial value of the follower force is p will leave an $O(\varepsilon)$-neighbourhood of the unstable solution when the follower force reaches a magnitude differing from $\Pi(p)$ by a quantity of the order of $\varepsilon|\ln \varepsilon|$ [4].

The behaviour of $\Pi(\cdot)$ in the interval I_1 is determined by the nature of the singularity of the function $\Psi(p)$ that obstructs the continuation of the family of arcs of level curves that cut the real axis twice and shrink to p_{cr} in the limit.

Let $p = u + iv$. The curves $\operatorname{Re} \Psi(u + iv) = \text{const}$ are phase trajectories of the system

$$du / ds = -\operatorname{Im} \lambda_1 / |\lambda_1|, \quad dv / ds = -\operatorname{Re} \lambda_1 / |\lambda_1| \tag{3.3}$$

where the independent variable s is a natural parameter of the curve. In complex form, system (3.3) may be written as $dp/ds = -i\bar{\lambda}_1/|\lambda_1|$.

By numerical integration of system (3.3), one can determine the position of the level curves $\operatorname{Re} \Psi(p) = \text{const}$ in the complex p-plane.

In the neighbourhood of branch points, the pattern of the level lines becomes fairly complicated (Section 6, Examples 2 and 3). Therefore, for a correct interpretation of the computational results, one must have some idea of the typical behaviour of the level curves in that case (Sections 4 and 5).

For some parameter values of Ziegler systems one can find approximate expressions for the roots of the characteristic equation of the linearized system and obtain explicit formulae for the input–output function (Section 6, Example 1).

4. BEHAVIOUR OF LEVEL CURVES IN THE NEIGHBOURHOOD OF A BRANCH POINT DUE TO THE MATRIX OF THE LINEARIZED SYSTEM HAVING MULTIPLE EIGENVALUES.

Consider the following system of differential equations depending on the complex parameter p

$$d\mathbf{z} / d\zeta = \mathbf{F}(\mathbf{z}, p), \quad \zeta \in \mathbb{C}^1, \quad \mathbf{z} \in \mathbb{C}^n \tag{4.1}$$

Let $\mathbf{z}_0(p)$ be a steady solution of system (4.1). Linearizing system (4.1) in the neighbourhood of this steady solution, we obtain

$$\frac{d\mathbf{w}}{d\zeta} = H_w(p)\mathbf{w}\left(\mathbf{w} = \mathbf{z} - \mathbf{z}_0(p),\ H_w(p) = \frac{\partial \mathbf{F}}{\partial \mathbf{z}}(\mathbf{z}_0(p), p)\right) \tag{4.2}$$

In the general case, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of the matrix $H_w(p)$ are simple and only at special, isolated values of p does one obtain matrices with one double eigenvalue associated with a Jordan cell of order 2 [12, p. 222]. Denote the set of such numbers p by Γ.

Suppose that $p_{cr} \in \Gamma$ we have $\lambda_1(p_{er}) = \lambda_2(p_{er}) = \lambda_0 \neq 0$. It follows from the normal form theorem for matrices dependent on a parameter [12, p. 217] that for p in some neighbourhood of p_{cr} there is a linear change of variables $\mathbf{v} = W(p)\mathbf{w}$, where $W(p)$ is a non-singular matrix whose coefficients are analytic functions of p, that transforms system (4.2) to a linear system with matrix

$$H_v(p) = \begin{Vmatrix} \Lambda_0 & 0 \\ 0 & \Lambda_1 \end{Vmatrix}$$

$$\Lambda_0 = \begin{Vmatrix} \lambda_0 & 1 \\ a(p - p_{er}) + O(|p - p_{er}|^2) & \lambda_0 + b(p - p_{er}) + O(|p - p_{er}|^2) \end{Vmatrix}$$

$$\Lambda_1 = \operatorname{diag}(\lambda_3(p), \ldots, \lambda_n(p))$$

The eigenvalues $\lambda_1(p)$, $\lambda_2(p)$ are the roots of the characteristic equation $\det(\Lambda_0 - \lambda E)$ and may be expressed in the neighbourhood of p_{cr} as

$$\lambda_{1,2}(p) = \lambda_0 \pm \sqrt{a(p - p_{er})} + O(|p - p_{er}|)$$

As p_{cr} describes a closed contour, $\lambda_1(p)$ and $\lambda_2(p)$ are interchanged.

To simplify the subsequent calculations, we change in (4.1) to a new independent variable $\xi = \lambda_0\zeta$. After this change the multiple eigenvalue of $H(p_{cr})$ will equal one.

Define a complex phase

$$\Psi(p) = \int_{p_{er}}^{p} \lambda_1(p')dp' = (p - p_{er}) + \frac{2}{3}\alpha(p - p_{er})^{3/2} + O(|p - p_{er}|^2)$$

where $\alpha = \sqrt{(a)}/\lambda_0$.

Let $\tilde{\Psi}(\rho) = \Psi(p_{er} + \rho^2)$, $\rho \in \mathbb{C}^1$. The transformation $p = p_{er} + \rho^2$ takes the level curves of the function $\operatorname{Re} \tilde{\Psi}(\rho)$ into those of $\operatorname{Re} \Psi(p)$.

The condition $\operatorname{Re} \tilde{\Psi}(\rho) = \text{const}$ may be rewritten as follows:

$$x^2 - y^2 + \frac{2}{3}[\alpha_1(x^3 - 3xy^2) + \alpha_2(y^3 - 3x^2y)] + O(x^4 + y^4) = \text{const}$$

$$x = \operatorname{Re}\rho, \quad y = \operatorname{Im}\rho, \quad \alpha_1 = \operatorname{Re}\alpha, \quad \alpha_2 = \operatorname{Im}\alpha$$

We shall assume below that $|\alpha_1| \neq |\alpha_2|$.

Solving the equation $\operatorname{Re} \tilde{\Psi}(x + iy) = 0$ for y, we obtain the equations of level curves passing through the point O as shown in Fig. 3(a) (curves aa′ and bb′)

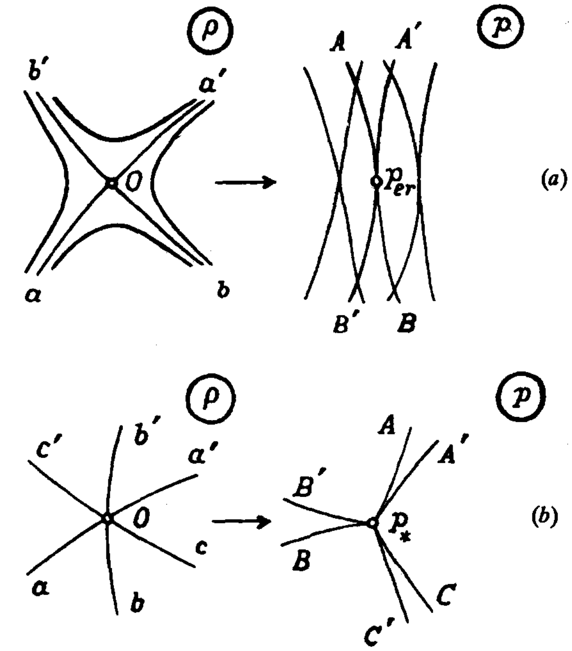

Fig. 3.

$$y=\pm x\mp\frac{2}{3}(\alpha_1\pm\alpha_2)x^2+O(x^3)$$

Let $\alpha_1 > |\alpha_2|$. The transformation $p = p_{cr} + \rho^2$ takes curves aa′ and bb′ into curves AA′ and BB′ with a cusp p_{cr} (Fig. 3a). The level curves Re $\tilde{\Psi}(\rho)$ = const in the sector *aOb* become level curves of Re $\Psi(p)$ = const in the sector $Ap_{cr}B$. Level curves in the sectors *aOb′* and *bOa′* go into level curves in the sectors $Ap_{cr}B'$ and $Bp_{cr}A'$ that intersect the arcs $p_{cr}A'$ and $p_{cr}B'$, respectively. Level curves in the sector *a′Ob′* go into level curves in the sector $A'p_{cr}B'$ that intersect the arcs $p_{cr}A$ and $p_{cr}B$.

Given a different relationship between α_1 and α_2, the position of the level curves of Re $\Psi(p)$ in the neighbourhood of p_{cr} is either that shown in Fig. 3(a) or its mirror image with respect to the vertical axis.

5. THE BEHAVIOUR OF LEVEL CURVES IN THE NEIGHBOURHOOD OF A BRANCH POINT OF A STEADY SOLUTION

Suppose that at $p = p_*$ we have $\lambda_1 = 0$. Applying a linear change of variables $\mathbf{w} = W(\mathbf{z}-\mathbf{z}_0(p))$, where W is some non-singular matrix with constant coefficients, combined with a change of the independent variable analogous to that used in Section 4, we transform system (4.1) to the system

$$\begin{aligned}\frac{dw_1}{d\xi}&=w_1^2-(p-p_*)+\sum_{i=1}^{n}\sum_{j=2}^{n}a_{ij}w_iw_j+O(|p-p_*||\mathbf{w}|,|\mathbf{w}|^3)\\ \frac{dw_k}{d\xi}&=\sum_{j=2}^{n}b_{kj}w_j+O(|p-p_*|,|\mathbf{w}|^2),\quad k=\overline{2,n}\end{aligned}\tag{5.1}$$

where the $(n-1)\times(n-1)$ matrix $\|b_{kj}\|$ is non-singular.

The steady solution $\mathbf{w}_0(p)$ of system (5.1) in the neighbourhood of p_* has the form

$$w_{10}(p)=\sqrt{p-p_*}+O(|p-p_*|)$$

$$w_{k0}(p)=c_k(p-p_*)+O(|p-p_*|^{3/2}),\quad k=\overline{2,n}$$

Thus, $p = p_*$ is a branch point of second order for the function $w_0(p)$.

The branching of the steady solution implies branching at $p = p_*$ of the eigenvalues of system (5.1), the latter being linearized in the neighbourhood of that solution

$$\lambda_1(p) = 2\sqrt{p - p_*} + \beta(p - p_*) + O(|p - p_*|^{3/2})$$

$$\lambda_k(p) = \lambda_k(p_*) + \gamma_k\sqrt{(p - p_*)} + O(|p - p_*|), \quad k = \overline{2,n}$$

Let us consider the behaviour in the complex plane of the level curves of the function Re $\Psi(p)$, where

$$\Psi(p) = \int_{p_*}^{p} \lambda_1(p')dp' = \frac{4}{3}(p - p_*)^{3/2} + \frac{\beta}{2}(p - p_*)^2 + O(|p - p_*|^{5/2})$$

Let $\tilde{\Psi}(\rho) = \Psi(p_* + \rho^2)$, $\tau \in \mathbb{C}^1$. The condition Re $\tilde{\Psi}(p)$ may be rewritten as follows:

$$\frac{8}{3}(x^3 - 3y^2x) + [\beta_1(x^4 - 6x^2y^2 + y^4) + 4\beta_2(xy^3 - x^3y)] + (|x|^5 + |y|^5) = \text{const}$$

$$x = \operatorname{Re}\rho, \quad y = \operatorname{Im}\rho, \quad \beta_1 = \operatorname{Re}\beta, \quad \beta_2 = \operatorname{Im}\beta$$

The level curves of Re $\tilde{\Psi} = 0$ are known as Stokes curves [13]. Solving Re $\tilde{\Psi}(x + iy) = 0$ for x, we obtain their equations

$$x = \beta_1 y^2 / 8 + O(|y|^3)$$

$$x = \pm\sqrt{3}y + \frac{1}{2}(\beta_1 \pm \sqrt{3}\beta_2)y^2 + O(|y|^3)$$

Let $\beta_1 > \sqrt{(3)}|\beta_2|$. Under the transformation $p = p_* + \rho^2$, the Stokes curves Oa, Ob, Oc, Oa', Ob', Oc' of the function $\tilde{\Psi}(p)$ pass into the Stokes curves $p_*A, p_*B, p_*C, p_*A', p_*B', p_*C'$ of the function $\Psi(p)$ (Fig. 3b). The level curves Re $\tilde{\Psi}(\rho)$ = const in the sector aOc' pass into level curves Re $\Psi(p)$ = const in the sector Ap_*C' that intersect the Stokes curves p_*C and p_*A'. Level curves in the sectors aOb, bOc, $a'Ob'$, $b'Oc'$ pass into level curves in the sectors Ap_*B, Bp_*C, $A'p_*B'$, $B'p_*C'$ that intersect the Stokes curves p_*B', p_*C', p_*A, p_*B, respectively. Level curves in the sector cOa' pass into level curves in the sector Cp_*A' that do not intersect Stokes curves.

The position of the level curves of Re $\Psi(p)$ in the neighbourhood of the point $p = p_*$ for other relative positions of β_1 and β_2 is the same as that shown in Fig. 3(b) rotated by an angle of $2\pi k/3$, where $k = 0$ or $k = \pm 1$.

6. EXAMPLES OF INPUT–OUTPUT FUNCTIONS FOR ZIEGLER SYSTEMS

Example 1. When a Ziegler system has sufficiently stiff hinges, the friction in which is also sufficiently high, the parameters of the problem satisfy the conditions

$$\varkappa \sim \varepsilon, \quad \varepsilon \ll \beta^{-1} \ll 1$$

Let us develop approximate formulae for the input–output function of such a system.

Setting $\kappa = \kappa_0\varepsilon$ to fix our ideas, we rewrite the equations of motion (1.2) in quasi-linear form

$$\dot{\mathbf{x}} = \Xi(p)\mathbf{x} + \Sigma_1 + \Sigma_2 \tag{6.1}$$

$$\mathbf{x} = \|\varphi_1, \varphi_2, \dot{\varphi}_1, \dot{\varphi}_2\|^{\mathrm{T}}, \quad \Sigma_1 = O(\varepsilon), \quad \Sigma_2 = O(|\mathbf{x}|^2)$$

$$\Xi(p) = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{Vmatrix}$$

$$h_{31} = -(3-p)/\mu,\quad h_{32} = (2-p)/\mu,\quad h_{33} = -3\beta/\mu,\quad h_{34} = 2\beta/\mu$$

$$h_{41} = 1 - h_{31},\quad h_{42} = -1 + p(1-\chi) - h_{32}$$

$$h_{43} = \beta - h_{33},\quad h_{44} = -1 - h_{34}$$

When $\varepsilon = 0$ system (6.1) always has a trivial steady solution $x \equiv 0$. This means that we are not concerned here with a case of general position [4].

The characteristic equation of system (6.1), linearized in the neighbourhood of the solution $x \equiv 0$ ($\varepsilon = 0$), is

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \tag{6.2}$$

$$a_0 = \mu,\quad a_1 = (5+\mu)\beta,\quad a_2 = \beta^2 + 5 + \mu - (\mu\chi_1 + 2)p$$

$$a_3 = (2 - 3\chi_1 p)\beta,\quad a_4 = 1 - (3-p)\chi_1 p,\quad \chi_1 = 1 - \chi$$

We have the following asymptotic formulae for the roots $\lambda_1, \ldots, \lambda_4$ of Eq. (6.2)

$$\lambda_{1,2} = -\beta^{-1}\left[\left(1 - \frac{3}{2}\chi_1 p\right) \pm i\sqrt{\left(1 - \frac{9}{4}\chi_1\right)\chi_1 p}\right] + O(\beta^{-2})$$

$$\lambda_{3,4} = \frac{\beta}{2\mu}[-(5+\mu) \pm \sqrt{25 + 6\mu + \mu^2}] + O(1)$$

If $\chi \leqslant 5/9$ ($\chi_1 \geqslant 4/9$), the stability loss is due to one of the roots of Eq. (6.2) crossing from the left half-plane into the right half-plane through zero, and there is no delay of the stability loss. We will therefore assume from now on that χ is a point in the interval (5/9, 1) ($\chi_1 \in$ (0. 4/9)).

The critical value of the follower force is found from the condition Re $\lambda_{1,2} = 0$

$$p_{cr}(\beta, \mu, \chi) = 2/3\chi_1 + O(\beta^{-1})$$

Using the asymptotic formulae for $\lambda_1(p)$ and p_{cr} in (3.1), we obtain an approximate expression for the complex phase

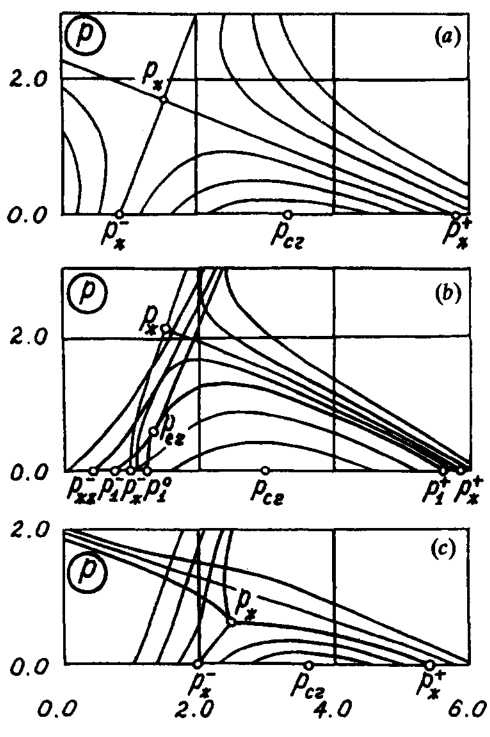

Fig. 4.

$$\Psi(p) = \beta^{-1}\left[\frac{1}{3\chi_1} - p + \frac{3}{4}\chi_1 p^2 - \right.$$

$$\left. -\frac{i}{2}\sqrt{\left(1-\frac{9}{4}\chi_1\right)\chi_1\left(p^2 - \frac{4}{9\chi_1^2}\right)}\right] \tag{6.3}$$

The error of formula (6.3) is at most $O(\beta^{-2})$.

The formula Re $\Psi(p)$ = const defines a family of hyperbolae in the complex plane. A general idea of the behaviour of the level curves Re $\Psi(p)$ = const may be derived from Fig. 4(a), which is shown for the case $\chi = 0.8$.

Note that since the steady solution does not bifurcate at the point where λ_1 vanishes, the pattern of the level curves is quite different from that shown in Fig. 3(b).

The point

$$p_* = \frac{3}{2} + i\sqrt{\frac{1}{\chi_1} - \frac{9}{4}}$$

at which $\lambda_1 = 0$, is a saddle point for Re $\Psi(p)$. The Stokes curves Re $[\Psi(p) - \Psi(p_*)] = 0$ intersect the real axis at points

$$p_*^{\pm}(\beta,\mu,\chi) = \frac{2}{3\chi_1} \pm \frac{2}{3}\sqrt{\frac{1}{\chi_1}\left(\frac{1}{\chi_1} - \frac{9}{4}\right)} + O(\beta^{-1})$$

The partition of the interval $I = [0, p_{cr}]$ described in Section 3 consists of the intervals $I_0 = (p_*, p_{cr})$ and $I_1 = [0, p_*]$.

We can write the input–output function in the interval I_0 as $\Pi(p) = 2p_{cr} - p$, with error $O(\beta^{-1})$.

In the interval I_1 the input–output function is given by $\Pi(p) \equiv p_*^+$. The point p_*^+ is a barrier point, which limits the time the phase trajectories of system (1.2) stay in the neighbourhood of an unstable equilibrium position. For solutions with the initial value of the follower force in I_1 the longest observed delay in stability loss is

$$\Delta(\chi) = p_*^+ - p_{cr} = \frac{2}{3}\sqrt{\frac{1}{1-\chi}\left(\frac{1}{1-\chi} - \frac{9}{4}\right)}$$

Note that $\Delta(\chi) \to 0$ as $\chi \to 5/9$; $\Delta(\chi) \to \infty$ as $\chi \to 1$.

Curve 1 in Fig. 5 is the graph of a function $\Pi(\cdot)$ constructed in accordance with the above approximation for a Ziegler system with parameters $\varkappa_0 = 2.0$, $\beta = 30.0$, $\delta = 1.5$, $\mu = 1.0$, $\chi = 0.8$. The small circles along the curve represent figures obtained by numerical integration of the equations of motion. In these computations, $\varepsilon = 10^{-3}$, $\varphi_1(p_i) = \varphi_2(p_i) = 0.5$, $\dot\varphi_1(p_i) = \dot\varphi_2(p_i) = 0$. The chosen value of p_0 was the magnitude of the tracking force at the first intersection (after the initial transient was completed) of a trajectory of system (1.2) with a sphere of radius $R = 0.33$ with centre at the point in the phase space corresponding to the steady solution of the instantaneous system. The results of the computations clearly confirm the estimate of the length of the stability loss delay obtained by using the input–output function.

Example 2. Let us return once more to a system with sufficiently stiff hinges ($\varkappa = \varkappa_0\varepsilon$). Let $\varkappa_0 = 0.5$, $\beta = 3.25$, $\delta = 1.5$, $\mu = 1.0$, $\chi = 0.85$. In that case the trivial solution $\mathbf{x} = 0$ of the equations of motion (6.1) with $\varepsilon = 0$ is stable when $p \leqslant p_{cr} = 3.06$.

Level curves of the function Re $\Psi(p)$ constructed by numerical integration of Eqs (3.3) are shown in Fig. 4(b).

As in Example 1, the point $p_* = 1.5 + 2.102i$ at which $\lambda_1 = 0$ is a regular point of $\Psi(p)$. The Stokes curves Re $[\Psi(p) - \Psi(p_*)] = 0$ cut the real axis at $p_*^- = 1.013$ and $p_*^+ = 5.857$.

The function $\Psi(p)$ experiences branching at the point $p_{cr} = 1.306 + 0.616i$, where λ_1 is a multiple root of the characteristic equation (Section 4). The level curves Re$[\Psi(p) - \Psi(p_{cr})] = 0$ cut the real axis at $p_1^- = 0.721$, $p_1^0 = 1.214$ and $p_1^+ = 5.622$.

There is no rigorous solution of the problem of constructing the input–output function $\Pi(\cdot)$ when the characteristic equation of the linearized system has multiple roots. We will determine the possible form of $\Pi(\cdot)$ by assuming, as in [6], that in such a situation the system has a certain property. Namely, as applied to system (6.1), when the follower force varies in the interval (p_{**}^-, p_*^+), a near-identical change of variables $\mathbf{x} \to \mathbf{x}_*$ exists that converts (6.1) into the system

$$\mathbf{x}_*^{\cdot} = [\Xi(p) + O(\varepsilon)]\mathbf{x}_* + O(|\mathbf{x}_*|^2) \tag{6.4}$$

which is satisfied when $\varepsilon \neq 0$ by the trivial solution $\mathbf{x}_* \equiv 0$. The number $p_{**}^- = 0.506$ satisfies the condition

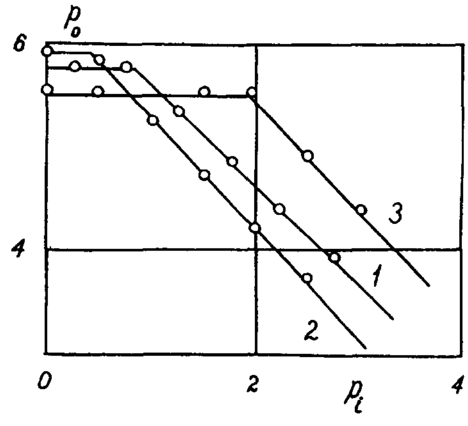

Fig. 5.

$$\mathrm{Re}\int_{p_{**}^-}^{p_*^+} \lambda_1(p')dp' = 0$$

In the stability domain, the speed at which a typical phase trajectory of system (6.4) approaches the origin is determined by the roots $\lambda_{1,2}$ of the characteristic equation (6.2) (it can be verified that $\lambda_4(p) < \lambda_3(p) < \mathrm{Re}\,\lambda_{1,2}(p)$). One can establish the following bound, characterizing the behaviour of a typical trajectory

$$C_1 \exp\left[\frac{1}{\varepsilon}\mathrm{Re}\int_{p_i}^{p} \lambda_1(p')dp'\right] \leq \frac{|\mathbf{x}_*(p)|}{|\mathbf{x}_*(p_i)|} \leq C_2 \exp\left[\frac{1}{\varepsilon}\mathrm{Re}\int_{p_i}^{p} \lambda_1(p')dp'\right] \tag{6.5}$$

provided that $0 < |\mathbf{x}_*(p_i)| \leq C_3$ (C_1, C_2 and C_3 are positive constants).

The bound (6.5) fails to hold for solutions whose speed of approach to the origin is determined by the roots $\lambda_3(p) < \lambda_4(p)$. The relative proportion of initial data generating such solutions is at most $O(\varepsilon^\eta)$, where η is an arbitrary positive number.

Inequality (6.5) implies the following formula for the input–output function $\Pi(\cdot)$ when $p_i \in I_0' = (p_{**}^-, p_{cr})$

$$\mathrm{Re}\int_{p_i}^{\Pi(p_i)} \lambda_1(p')dp' = 0, \quad \Pi(p_i) > p_{cr} \tag{6.6}$$

Indeed, if condition (6.5) is satisfied, the phase trajectory, having entered an ε-neighbourhood of the origin at $p = p_i + O(\varepsilon\,|\ln\varepsilon\,|)$, leaves that neighbourhood at $p = \Pi(p_i) + O(\varepsilon\,|\ln\varepsilon\,|)$.

Formula (6.6) may be transformed to the form of (3.2).

In the interval $I_1' = [0, p_{**}^-]$, the function $\Pi(\cdot)$ of system (6.1) takes the constant value p_*^+.

The results of [4] enable us to justify the above definition of input–output-function only in the interval $I_0 = (p_1^-, p_{cr}) \subset I_0'$. Note that the point $p_1^- \in I_0'$, which is joined to p_{cr} by an arc of a level curve of the function $\mathrm{Re}\,\Psi(p)$, is not a singular point of the function $\Pi(\cdot)$ given by (6.6).

Curve 2 in Fig. 5 is the graph of the proposed input–output function for the system we are studying. As in the previous example, the small circles represent results of a numerical computation of stability loss delay ($\varepsilon = 10^{-2}$).

Example 3. Consider a Ziegler system with the parameters specified in Section 2. The level curves of the function $\mathrm{Re}\,\Psi(p)$ for that system are shown in Fig. 4(c).

The point $p_* = 2.457 + 0.627i$ at which $\lambda_1 = 0$ is a branch point for $\mathrm{Re}\,\Psi(p)$, as shown in Fig. 5. The Stokes curves $\mathrm{Re}\,[\Psi(p) - \Psi(p_*)] = 0$ intersect the real axis at $p_*^- = 1.968$ and $p_*^+ = 5.491$.

In the interval $I_0 = (p_*^-, p_{cr})$ the input–output function in the case considered here maps a point p onto a point $\Pi(p)$ of the interval (p_{cr}, p_*^+) joined to p by an arc of a level curve of $\mathrm{Re}\,\Psi(p)$. In the interval $I_1 = [0, p_*^-)$ we have $\Pi(p) = p_*^+$ [5].

A graph of the input–output function is shown in Fig. 5 (curve 3). The small circles are results of a numerical computation of the stability loss delay ($\varepsilon = 10^{-3}$).

Note that the full pattern of the level curves in the neighbourhood of a branch point is not needed to construct the input–output-function. In the notation of Section 5, the Stokes curves that cut the real axis at points p_*^- and p_*^+ will be p_*B' and p_*C', respectively.

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